

Vacillating Hecke Tableaux and Linked Partitions

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Abstract

We introduce the structure of vacillating Hecke tableaux, and establish a one-to-one correspondence between vacillating Hecke tableaux and linked partitions by using the Hecke insertion algorithm developed by Buch, Kresch, Shimozono, Tamvakis and Yong. Linked partitions arise in free probability theory. Motivated by the Hecke insertion algorithm, we define a Hecke diagram as a Young diagram possibly with a marked corner. A vacillating Hecke tableau is defined as a sequence of Hecke diagrams subject to certain addition and deletion of rook strips. The notion of a rook strip was introduced by Buch in the study of the Littlewood-Richardson rule for stable Grothendieck polynomials. A rook strip is a skew Young diagram with at most one square in each row and column. We show that the crossing number and the nesting number of a linked partition can be determined by the maximal number of rows and the maximal number of columns of the diagrams in the corresponding vacillating Hecke tableau. The proof relies on a theorem due to Thomas and Yong concerning the lengths of the longest strictly increasing and the longest strictly decreasing subsequences in a word. This implies that the crossing number and the nesting number have a symmetric joint distribution over linked partitions, confirming a conjecture of de Mier. We also prove a conjecture of Kim which states that the crossing number and the nesting number have a symmetric joint distribution over the front representations of partitions.

Keywords: Linked partition, crossing, nesting, the Hecke insertion algorithm

AMS Classifications: 05A05, 05A17, 05C30

1 Introduction

The Hecke insertion algorithm was developed by Buch, Kresch, Shimozono, Tamvakis and Yong [2] in order to expand a stable Grothendieck polynomial in terms of stable Grothendieck polynomials indexed by integer partitions. Stable Grothendieck polynomials were defined by Fomin and Kirillov [9] as a limit of ordinary Grothendieck polynomials of Lascoux and Schützenberger [13]. For a permutation π , let $G_\pi = G_\pi(x_1, x_2, \dots)$ be the stable Grothendieck polynomial indexed by π . Fomin and Kirillov [9] showed that G_π can be explained as a weighted counting of compatible pairs. On the other hand, Buch [1] showed that

$$G_\pi = \sum_{\lambda} c_{\pi, \lambda} G_{\pi_\lambda}, \quad (1.1)$$

where the sum ranges over integer partitions, and for a partition λ , $c_{\pi, \lambda}$ is an integer and π_λ is a permutation determined by λ . Buch [1] proved that the polynomial G_{π_λ} can be interpreted combinatorially in terms of set-valued tableaux of shape λ . A set-valued tableau of shape λ is an assignment of nonempty sets of positive integers to the squares of the Young diagram of λ such that the sets are weakly increasing along each row and strictly increasing along each column, where for two sets A and B of integers, the relation $A \leq B$ (resp., $A < B$) means that $\max(A) \leq \min(B)$ (resp., $\max(A) < \min(B)$).

By developing the Hecke insertion algorithm, Buch et al. [2] constructed a bijection between compatible pairs and pairs (T, U) , where T is an increasing tableau and U is a set-valued tableau of the same shape as T . An increasing tableau of shape λ is an assignment of positive integers to the squares of λ such that the numbers are strictly increasing in each row and each column. Using this bijection, Buch et al. [2] showed that the coefficient $c_{\pi, \lambda}$ in (1.1) equals up to a sign the number of increasing tableaux of shape λ satisfying certain conditions. The Hecke algorithm can be viewed as an extension of the Robinson-Schensted algorithm [15, 17] and the Edelman-Greene algorithm [8].

In this paper, we introduce the structure of vacillating Hecke tableaux, and establish a one-to-one correspondence between vacillating Hecke tableaux and linked partitions by using the Hecke algorithm. Motivated by the Hecke algorithm, we define a Hecke diagram as a Young diagram possibly with a marked corner. A vacillating Hecke tableau is a sequence of Hecke diagrams subject to certain addition and deletion of rook strips. The notion of a rook strip was introduced by Buch [1] in the study of the Littlewood-Richardson rule for stable Grothendieck polynomials. More precisely, a rook strip is a skew Young diagram with at most one square in each row and column. When the Hecke diagrams are restricted to Young diagrams and the rook strips are restricted to single squares, a vacillating Hecke tableau specializes to an ordinary vacillating tableaux due to Chen, Deng, Du, Stanley and Yan [3].

We show that the crossing number and the nesting number of a linked partition can be determined by the maximal number of rows and the maximal number of columns of diagrams in the corresponding vacillating Hecke tableau. The proof relies on a theorem due to Thomas and Yong [19] which states that the insertion tableau of a word generated

by the Hecke algorithm determines the lengths of the longest strictly increasing and strictly decreasing subsequences in the word.

As a consequence, we show that the crossing number and the nesting number have a symmetric joint distribution over linked partitions. This confirms a conjecture posed by de Mier [6]. We also show that the crossing number and the nesting number have a symmetric joint distribution over the front representations of set partitions. This proves a conjecture of Kim [10].

The notion of a linked partition was introduced by Dykema [7] in the study of unsymmetrized T-transforms in free probability theory. Let $[n] = \{1, 2, \dots, n\}$. Dykema [7] showed that noncrossing linked partitions of $[n+1]$ are counted by the n -th large Schröder number. Chen, Wu and Yan [4] found a combinatorial interpretation of this fact by establishing a bijection between noncrossing linked partitions of $[n+1]$ and Schröder paths of length n .

A linked partition of $[n]$ is a collection of nonempty subsets B_1, B_2, \dots, B_k of $[n]$, called blocks, such that the union of B_1, B_2, \dots, B_k is $[n]$ and any two distinct blocks are nearly disjoint. Two distinct blocks B_i and B_j are said to be nearly disjoint if for any $t \in B_i \cap B_j$, one of the following conditions holds:

- (1) $t = \min(B_i)$, $|B_i| > 1$, and $t \neq \min(B_j)$,
- (2) $t = \min(B_j)$, $|B_j| > 1$, and $t \neq \min(B_i)$.

The linear representation of a linked partition was defined by Chen, Wu and Yan [4]. For a linked partition P of $[n]$, list the n vertices $1, 2, \dots, n$ in increasing order on a horizontal line. For a block $B_i = \{a_1, a_2, \dots, a_m\}$ with $m \geq 2$ and $a_1 < a_2 < \dots < a_m$, draw an arc from a_1 to a_j for $j = 2, \dots, m$. For example, the linear representation of the linked partition $\{\{1, 3, 5\}, \{2, 6, 10\}, \{4\}, \{5, 8, 9\}, \{6, 7\}\}$ is illustrated in Figure 1.1. By definition, it is easily checked that the linear representation of a linked partition of

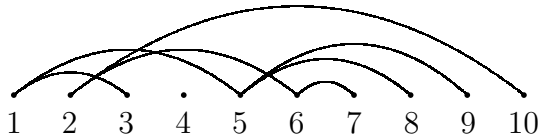


Figure 1.1: The linear representation of a linked partition.

$[n]$ is a simple graph on $[n]$ such that for each vertex i there is at most one vertex j with $1 \leq j < i$ that is connected to i , and vice versa.

The crossing number and the nesting number of a linked partition P are defined based on k -crossings and k -nestings in the linear representation of P , where k is a positive integer. We use a pair (i, j) with $i < j$ to denote an arc in the linear representation

of P , and we call i and j the left-hand endpoint and the right-hand endpoint of (i, j) , respectively. We say that k arcs $(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)$ of P form a k -crossing if

$$i_1 < i_2 < \dots < i_k < j_1 < j_2 < \dots < j_k,$$

and form a k -nesting if

$$i_1 < i_2 < \dots < i_k < j_k < \dots < j_2 < j_1.$$

The crossing number $\text{cr}(P)$ of P is defined as the maximal number k such that P has a k -crossing. Similarly, the nesting number $\text{ne}(P)$ of P is the maximal number k such that P has a k -nesting. For example, for the linked partition in Figure 1.1, we have $\text{cr}(P) = 2$ and $\text{ne}(P) = 3$.

de Mier [6] posed the following conjecture and showed that it holds for $i = 1$.

Conjecture 1.1 (de Mier [6]) *For any positive integers i and j , the number of linked partitions P of $[n]$ with $\text{cr}(P) = i$ and $\text{ne}(P) = j$ equals the number of linked partitions P of $[n]$ with $\text{cr}(P) = j$ and $\text{ne}(P) = i$.*

Kim [10] posed a conjecture on the joint distribution of the crossing number and the nesting number of front representations of set partitions. A set partition of $[n]$ is a collection of mutually disjoint nonempty subsets whose union is $[n]$. Clearly, a set partition of $[n]$ is a linked partition of $[n]$ such that any two distinct blocks are disjoint. When P is a set partition, the linear representation of P is called the front representation of P by Kim [10].

For example, Figure 1.2 is the front representation of the set partition

$$\{\{1, 3, 5, 8\}, \{2, 6, 9\}, \{4\}, \{7, 10\}\}.$$

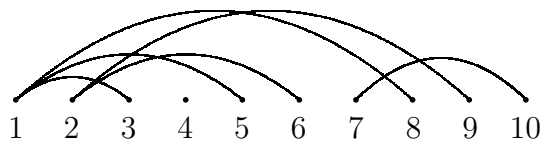


Figure 1.2: The front representation of a set partition.

Conjecture 1.2 (Kim [10]) *For any positive integers i and j , the number of front representations of set partitions P of $[n]$ with $\text{cr}(P) = i$ and $\text{ne}(P) = j$ equals the number of front representations of set partitions P of $[n]$ with $\text{cr}(P) = j$ and $\text{ne}(P) = i$.*

We remark that the crossing number and the nesting number of a set partition have been defined based on the standard representation, see [3]. The distributions of the crossing number and the nesting number have been extensively studied in the context of fillings of diagrams, see, for example, [5, 12, 14, 16].

This paper is organized as follows. In Section 2, we present the definition of a vacillating Hecke tableau. In Section 3, we give an overview of the Hecke algorithm developed by Buch et al. [2], as well as some properties of this algorithm. Section 4 provides a bijection between vacillating Hecke tableaux and linked partitions based on the Hecke algorithm. In particular, we show that the crossing number and the nesting number of a linked partition are determined by the maximal number of rows and the maximal number of columns of diagrams in the corresponding vacillating Hecke tableau. As consequences, we confirm Conjecture 1.1 and Conjecture 1.2.

2 Vacillating Hecke Tableaux

In this section, we define a vacillating Hecke tableau as a sequence of Hecke diagrams subject to certain addition and deletion of rook strips. We first give the notion of a Hecke diagram. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ be a partition of a positive integer n , that is, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$ and $\lambda_1 + \lambda_2 + \dots + \lambda_\ell = n$. The Young diagram of λ is a left-justified array of squares with λ_i squares in row i .

A Hecke diagram is defined as a Young diagram possibly with a marked corner. For example, Figure 2.1 gives the Hecke diagrams whose underlying Young diagram is $(4, 4, 2, 1)$, where we use a bullet to indicate a marked corner. We call a Hecke diagram

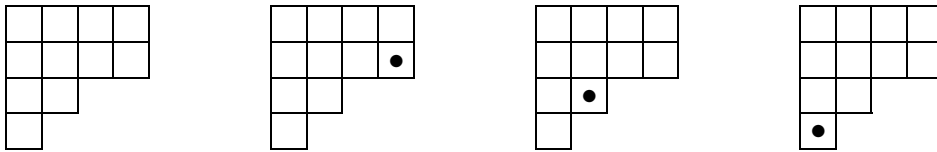


Figure 2.1: Hecke diagrams with underlying Young diagram $(4, 4, 2, 1)$.

an ordinary diagram if it does not have a marked corner, and a marked diagram if it has a marked corner. When λ is a marked diagram with a marked corner c , we also write λ as a pair (μ, c) , where μ is the underlying Young diagram of λ .

To define a vacillating Hecke tableau, we need the notion of a rook strip introduced by Buch [1]. For two Young diagrams λ and μ such that μ is contained in λ , the skew diagram λ/μ is the collection of squares of λ that are outside μ . A rook strip is a skew diagram with at most one square in each row and column. For example, in Figure 2.2, the skew diagram (a) is a rook strip, but (b) is not a rook strip.

A vacillating Hecke tableau of empty shape and length $2n$ is defined to be a sequence

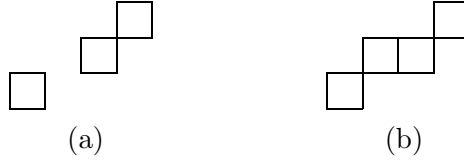


Figure 2.2: Examples of skew diagrams.

$(\lambda^0, \lambda^1, \dots, \lambda^{2n})$ of Hecke diagrams such that

- (i) $\lambda^0 = \lambda^{2n} = \emptyset$, and for $1 \leq i \leq n$, λ^{2i-1} is an ordinary diagram;
 - (ii) If λ^{2i} is an ordinary diagram, then λ^{2i-1} is an ordinary diagram contained in λ^{2i} such that $\lambda^{2i-1} = \lambda^{2i}$ or $\lambda^{2i}/\lambda^{2i-1}$ is a rook strip, and λ^{2i+1} is an ordinary diagram contained in λ^{2i} such that $\lambda^{2i+1} = \lambda^{2i}$ or $\lambda^{2i}/\lambda^{2i+1}$ is a square;
- If $\lambda^{2i} = (\mu, c)$ is a marked diagram, then λ^{2i-1} is an ordinary diagram contained in μ such that $\lambda^{2i-1} = \mu$ or μ/λ^{2i-1} is a rook strip, and $\lambda^{2i+1} = \mu$.

As an example, Figure 2.3 illustrates a vacillating Hecke tableau of empty shape and length 14.

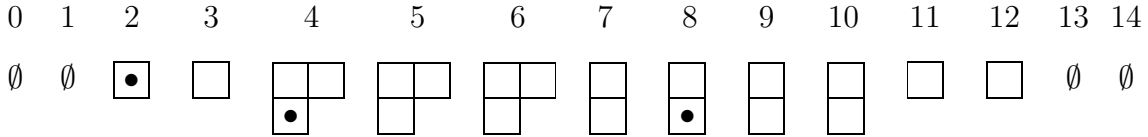


Figure 2.3: A vacillating Hecke tableau of empty shape and length 14.

Notice that when the diagrams of a vacillating Hecke tableau are restricted to ordinary diagrams and rook strips are replaced by single squares, a vacillating Hecke tableau reduces to an ordinary vacillating tableau [3].

3 The Hecke insertion algorithm

In this section, we shall give an overview of the Hecke insertion algorithm developed by Buch et al. [2], as well as a theorem due to Thomas and Yong [19] which states that the insertion tableau of a word generated by the Hecke insertion algorithm determines the lengths of the longest strictly increasing and strictly decreasing subsequences in the word. We also prove a property concerning the insertion tableau of a word, which will be used in proof of the main result of this paper, as given in the next section.

The Hecke algorithm is a procedure to insert a positive integer into an increasing tableau, which leads to a representation of a word by an increasing tableau and a set-valued tableau. Let λ be a Young diagram. An increasing tableau T of shape λ is an assignment of positive integers to the squares of λ such that the numbers are strictly increasing in each row and column. Suppose that U is the tableau obtained from T by inserting a positive integer x . Then U is either of the same shape as T or it has an extra square compared with T . In the case when U has the same shape as T , it also contains a special corner where the algorithm terminates and this corner needs to be recorded. A parameter $\alpha \in \{0, 1\}$ is used to distinguish these two cases. Thus the output of the Hecke algorithm when applied to T is a triple (U, c, α) , where c is a corner of U .

The Hecke algorithm can be described as follows. Assume that T is an increasing tableau and x is a positive integer. To insert x into T , we begin with the first row of T . Roughly speaking, an element in this row may be bumped out and then inserted into the next row. The process is repeated until no more element is bumped out. More precisely, let R be the first row of T . We have the following two cases.

Case 1: The integer x is larger than or equal to all entries in R . If adding x as a new square to the end of R results in an increasing tableau, then U is the resulting tableau, c is the corner where x is added. We set $\alpha = 1$ to signify that the corner c is outside the shape of T , and the process terminates. If adding x as a new square to the end of R does not result in an increasing tableau, then let $U = T$, and c be the corner at the bottom of the column of U containing the rightmost square of R . In this case, we set $\alpha = 0$ to indicate that the corner c is inside the shape of T , and the process terminates.

Case 2: The integer x is strictly smaller than some element in R . Let y be the leftmost entry in R that is strictly larger than x . If replacing y by x results in an increasing tableau, then y is bumped out by x and y will be inserted into the next row. If replacing y by x does not result in an increasing tableau, then keep the row R unchanged and the element y will also be inserted into the next row.

We can iterate the above process to insert the element y into the next row, still denoted by R . Finally, we get the output (U, c, α) of the insertion algorithm, and we write $U = (T \xleftarrow{H} x)$ and $(U, c, \alpha) = H(T, x)$.

We give two examples to demonstrate the two cases $\alpha = 0$ and $\alpha = 1$ of the insertion algorithm. Let T be an increasing tableau of shape $(4, 3, 2, 2)$ as given in Figure 3.1. Let

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 5 & \\ \hline 4 & 5 & & \\ \hline 5 & 7 & & \\ \hline \end{array}$$

Figure 3.1: An increasing tableau of shape $(4, 3, 2, 2)$.

$x = 1$. The process to insert x into T is illustrated in Figure 3.2, where the an element

in boldface represents the entry that is bumped out and is to be inserted into the next row. We see that the resulting tableau U has one more square than T , and so we have $\alpha = 1$.

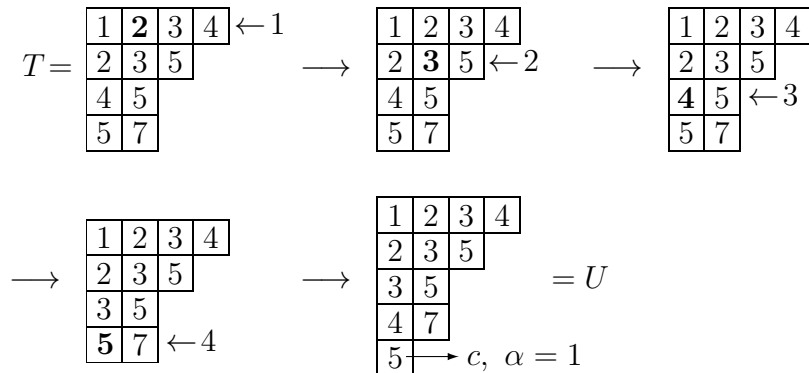


Figure 3.2: An example of the Hecke insertion algorithm for $\alpha = 1$.

For $x = 3$, we find that the resulting tableau U has the same shape as T , and so we have $\alpha = 0$, see Figure 3.3.

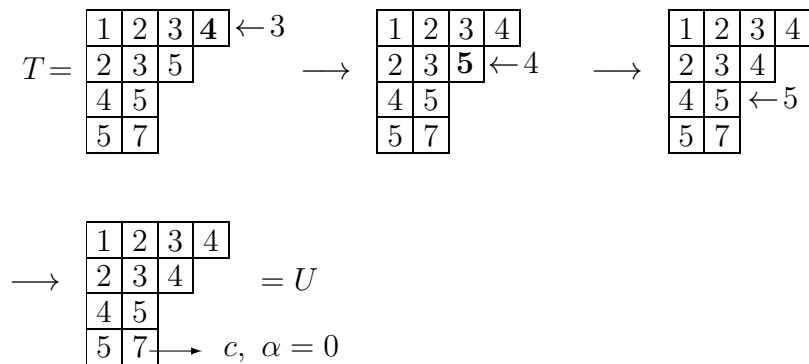


Figure 3.3: An example of the Hecke insertion algorithm for $\alpha = 0$.

The Hecke algorithm is reversible, see Buch et al. [2]. In other words, give an increasing tableau U , a corner c of U , and the value of α , there exist a unique increasing tableau T and a unique positive integer x such that $U = (T \xleftarrow{H} x)$.

Thomas and Yong [19] showed that the Hecke algorithm can be used to determine the lengths of the longest strictly increasing and strictly decreasing subsequences of a word. Let $w = w_1 w_2 \cdots w_n$ be a word of positive integers. A subword of $w = w_1 w_2 \cdots w_n$ is a subsequence $w_{i_1} w_{i_2} \cdots w_{i_k}$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. A subword $w_{i_1} w_{i_2} \cdots w_{i_k}$ is said to be strictly increasing if $w_{i_1} < w_{i_2} < \cdots < w_{i_k}$, and strictly decreasing if

$w_{i_1} > w_{i_2} > \cdots > w_{i_k}$. Let $\text{is}(w)$ (resp., $\text{de}(w)$) denote the length of the longest strictly increasing (resp., strictly decreasing) subwords of w . As shown by Thomas and Yong, $\text{is}(w)$ and $\text{de}(w)$ are determined by the shape of the insertion tableau of w . The insertion tableau of w is defined by

$$(\cdots((\emptyset \xleftarrow{\text{H}} w_1) \xleftarrow{\text{H}} w_2) \xleftarrow{\text{H}} \cdots) \xleftarrow{\text{H}} w_n.$$

For example, let $w = 21131321$. The construction of the insertion tableau of w is given in Figure 3.4.

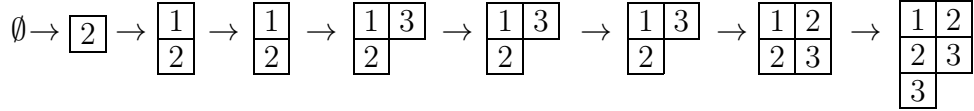


Figure 3.4: The insertion tableau of $w = 21131321$.

For an increasing tableau T , let $c(T)$ and $r(T)$ denote the number of columns and the number of rows of T , respectively. Using the jeu de taquin algorithm for increasing tableaux developed in [18], Thomas and Yong [19] established the following relation.

Theorem 3.1 *Let w be a word of positive integers, and T be the insertion tableau of w . Then $\text{is}(w) = c(T)$ and $\text{de}(w) = r(T)$.*

We observe the following property of the insertion tableau.

Proposition 3.2 *Let $w = w_1w_2\cdots w_n$ be a word of positive integers, and k be the maximal element appearing in w . Let $w' = a_1a_2\cdots a_m$ be the word obtained from w by deleting the elements equal to k . Assume that T is the insertion tableau of w and T' is the insertion tableau of w' . Then T' is obtained from T by deleting the squares occupied with k .*

Proof. Let Q denote the increasing tableau obtained from T by deleting the squares occupied with the maximal element k . We use induction to prove that $T' = Q$. The claim is obvious when $n = 1$. We now assume that $n > 1$ and that the claim holds for $n - 1$. Let P be the insertion tableau of $w_1w_2\cdots w_{n-1}$. Here are two cases.

Case 1: $w_n = k$. By the induction hypothesis, T' is obtained from P by deleting the squares occupied with k . On the other hand, since $w_n = k$ is the maximal element of w , we see that $T = P$ or T is obtained from P by adding a square filled with k at the end of the first row. This yields that $T' = Q$.

Case 2: $w_n < k$. Let U be the insertion tableau of $a_1\cdots a_{m-1}$. By the induction hypothesis, U is obtained from P by deleting the squares occupied with k . In the

process of inserting w_n into P , if no entry equal to k is bumped out and is inserted into the next row, then it is clear that $T' = Q$.

Otherwise, there is a unique entry k in P that is bumped out and is inserted into the next row. Let c be the square of P occupied with this entry. Note that c is a corner of P since P is increasing and k is a maximal entry. Keep in mind that U is obtained from P by deleting the squares occupied with k . Since $T = (P \xleftarrow{H} w_n)$ and $T' = (U \xleftarrow{H} w_n)$, for any square C in U , the entry of T' in C equals the entry of T in C . Consequently, to verify $T' = Q$, it suffices to consider the entry of T in the corner c . Assume that this entry is equal to i . Here are two subcases.

Case 2.1: $i = k$. In this case, T' has the same shape as U . On the other hand, any square of T outside U is occupied with k . So we have $T' = Q$.

Case 2.2: $i < k$. In this case, T' has the extra corner c compared with U . By the construction of the Hecke algorithm, we see that the entry of T' in the corner c also equals i . Notice also that except for the corner c , any square of T outside U is occupied with k . So we are led to $T' = Q$. This completes the proof. ■

For example, let $w = 21131321$. Then we have $w' = 211121$. The insertion tableau T of w is given in Figure 3.4. Meanwhile, the insertion tableau T' of w' is constructed in Figure 3.5, which coincides with the tableau obtained from T by deleting the squares occupied with 3.

$$\emptyset \rightarrow \boxed{2} \rightarrow \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}$$

Figure 3.5: The insertion tableau of $w' = 211121$.

4 Vacillating Hecke tableaux and linked partitions

In this section, we provide a bijection between vacillating Hecke tableaux and linked partitions. We prove that the crossing number and the nesting number of a linked partition can be determined by the maximal number of rows and the maximal number of columns of diagrams in the corresponding vacillating Hecke tableau. As a consequence, we show that the crossing number and the nesting number have a symmetric joint distribution over linked partitions with fixed left-hand endpoints and right-hand endpoints. This leads to a proof of Conjecture 1.1. Specializing the bijection to linked partitions containing no vertex that is both a left-hand endpoint and a right-hand endpoint, we confirm Conjecture 1.2.

To describe our bijection, by a Hecke tableau we mean an increasing tableau possibly with a marked corner. In other words, a Hecke tableau is an increasing tableau whose

shape is a Hecke diagram. Let λ be a Hecke diagram, and let T be a Hecke tableau of shape λ . When $\lambda = (\mu, c)$ is a marked diagram, we also express T by a pair (T', c) , where T' is the underlying increasing tableau of T .

Let V_{2n} be the set of vacillating Hecke tableaux of empty shape and length $2n$. We now give a description of a bijection ϕ from V_{2n} to the set of linked partitions of $[n]$.

Let V be a vacillating Hecke tableau $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset)$ of empty shape and length $2n$. First, we recursively define a sequence $(E_0, T_0), (E_1, T_1), \dots, (E_{2n}, T_{2n})$, where for $0 \leq i \leq 2n$, E_i is a set of edges and T_i is a Hecke tableau of shape λ^i . Let $E_0 = \emptyset$, and let T_0 be the empty tableau. Assume that $i \geq 1$. If $\lambda^i = \lambda^{i-1}$, then we let $(E_i, T_i) = (E_{i-1}, T_{i-1})$. If $\lambda^i \neq \lambda^{i-1}$, (E_i, T_i) is constructed according to the parity of i .

Case 1: i is odd. Let $i = 2k - 1$. By the definition of a vacillating Hecke tableau, λ^i is a diagram without any marked corner. Here are two subcases according to whether λ^{i-1} is an ordinary diagram.

Case 1.1: λ^{i-1} is an ordinary diagram. Then λ^i is obtained from λ^{i-1} by deleting a corner c . Setting $\alpha = 1$, there are a unique increasing tableau T and a unique positive integer j such that $(T_{i-1}, c, \alpha) = H(T, j)$. Let $T_i = T$ and define E_i to be the set obtained from E_{i-1} by adding the edge (j, k) .

Case 1.2: $\lambda^{i-1} = (\mu, c)$ is a marked diagram. So we have $\lambda^i = \mu$. Setting $\alpha = 0$, there are a unique increasing tableau T and a unique positive integer j such that $(T_{i-1}, c, \alpha) = H(T, j)$. Let $T_i = T$ and define E_i to be the set obtained from E_{i-1} by adding the edge (j, k) .

Case 2: i is even. Let $i = 2k$. We set $E_i = E_{i-1}$. To define T_i , there are two subcases according to whether λ^i is an ordinary diagram.

Case 2.1: λ^i is an ordinary diagram. Then λ^i/λ^{i-1} is a rook strip. Define T_i to be the tableau obtained from T_{i-1} by filling the squares of λ^i/λ^{i-1} with k .

Case 2.2: $\lambda^i = (\mu, c)$ is a marked diagram. Then μ/λ^{i-1} is a rook strip. Let T be the tableau of shape μ that is obtained from T_{i-1} by filling the squares of μ/λ^{i-1} with k . Define $T_i = (T, c)$.

Finally, we define $\phi(V)$ to be the diagram with n vertices $1, 2, \dots, n$ listed on a horizontal line such that there is an arc connecting j and k with $j < k$ if and only if (j, k) is an edge in E_{2n} . By the above construction, for each vertex $k \in [n]$, there is at most one vertex j with $j < k$ such that (j, k) is an arc in $\phi(V)$. Thus $\phi(V)$ is a linked partition of $[n]$.

Figure 4.1 gives an illustration of the map ϕ when applied to the vacillating Hecke tableau in Figure 2.3, where an entry in boldface indicates a marked corner of a Hecke tableau.

It can be checked that ϕ is reversible, and hence it is a bijection. Let P be a linked

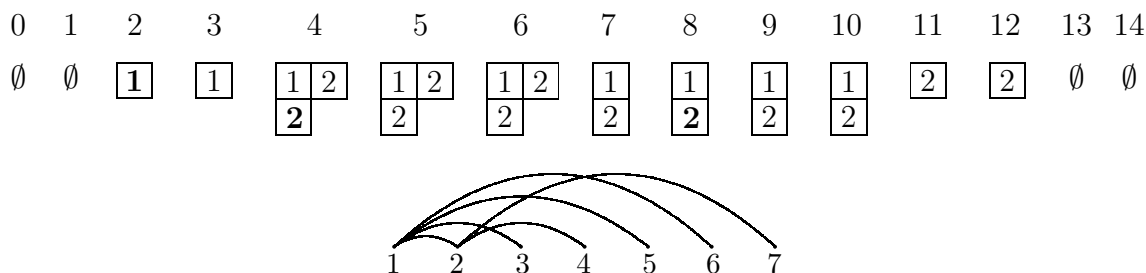


Figure 4.1: An illustration of the bijection ϕ .

partition of $[n]$. To recover the corresponding vacillating Hecke tableau, we first construct a sequence $(T_0, T_1, \dots, T_{2n})$ of Hecke tableaux. Let T_{2n} be the empty tableau. Suppose that T_{2i} has been constructed, where $1 \leq i \leq n$. We proceed to construct T_{2i-1} and T_{2i-2} . To obtain T_{2i-1} , we have two cases.

Case 1: The shape of T_{2i} is an ordinary diagram. Let T_{2i-1} be the tableau obtained from T_{2i} by deleting the squares (if any) filled with i .

Case 2: The shape of T_{2i} is a marked diagram. Assume that $T_{2i} = (T, c)$. Let T_{2i-1} be the tableau obtained from T by deleting the squares (if any) filled with i .

Now the Hecke tableau T_{2i-2} is obtained from T_{2i-1} . If i is not a right-hand endpoint of any arc of P , then we set $T_{2i-2} = T_{2i-1}$. Otherwise, there is a unique arc (j, i) with $j < i$ of P . Assume that $(U, c, \alpha) = H(T_{2i-1}, j)$. We set $T_{2i-2} = U$ if $\alpha = 1$, and set $T_{2i-2} = (U, c)$ if $\alpha = 0$.

Let λ^i be the shape of T_i . Finally, the vacillating Hecke tableau $\phi^{-1}(P)$ is given by

$$(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset).$$

The following theorem shows that the crossing number and the nesting number of a linked partition are determined by the diagrams in the corresponding vacillating Hecke tableau. For a vacillating Hecke tableau V , let $r(V)$ be the most number of rows in any diagram λ^i of V . Similarly, let $c(V)$ be the most number of columns in any diagram λ^i of V . We have the following relations.

Theorem 4.1 *Let V be a vacillating Hecke tableau in V_{2n} , and let $P = \phi(V)$. Then we have $c(V) = \text{ne}(P)$ and $r(V) = \text{cr}(P)$.*

Proof. Let $V = (\lambda^0, \lambda^1, \dots, \lambda^{2n})$, and let $(T_0, T_1, \dots, T_{2n})$ be the sequence of Hecke tableaux in the construction of ϕ . For $0 \leq i \leq 2n$, let U_i be the underlying increasing tableau of T_i . We shall generate a sequence $(w^{(0)}, w^{(1)}, \dots, w^{(2n)})$ of words such that U_i is the insertion tableau of $w^{(i)}$.

Let $w^{(2n)}$ be the empty word. Suppose that $w^{(i)}$ has been constructed. Then $w^{(i-1)}$ is constructed as follows. If $T_{i-1} = T_i$, then we let $w^{(i-1)} = w^{(i)}$. If $T_{i-1} \neq T_i$, we have two cases.

Case 1: i is odd. Let $i = 2k - 1$. By the construction of ϕ , we see that U_{i-1} is obtained from U_i by inserting a unique integer j . Define $w^{(i-1)} = w^{(i)} j$;

Case 2: i is even. Let $i = 2k$. Again, by the construction of ϕ , we find that U_{i-1} is obtained from U_i by deleting the squares (if any) filled with k . Define $w^{(i-1)}$ to be the word obtained from $w^{(i)}$ by removing the elements (if any) equal to k .

We proceed by induction to show that U_i is the insertion tableau of $w^{(i)}$. The claim is obvious for $i = 2n$. Assume that the claim is true for i , where $1 \leq i \leq 2n$. We wish to prove that it holds for $i - 1$. If $T_{i-1} = T_i$, then the claim is evident. Let us now consider the case when $T_{i-1} \neq T_i$. If $w^{(i-1)}$ is generated according to Case 1, then the claim follows directly from the construction of ϕ . If $w^{(i-1)}$ is generated according to Case 2, then the claim is a consequence of Proposition 3.2. This proves the claim.

Combining the above claim and Theorem 3.1, we obtain that

$$c(V) = \max \{ \text{is}(w^{(i)}) \mid 0 \leq i \leq 2n \}$$

and

$$r(V) = \max \{ \text{de}(w^{(i)}) \mid 0 \leq i \leq 2n \}.$$

It remains to show that P has a k -crossing (resp., k -nesting) if and only if there exists a word $w^{(i)}$ that contains a strictly decreasing (resp., increasing) subword of length k . We shall only give the proof of the statement concerning the relationship between a k -crossing and a strictly decreasing subword of length k . The same argument applies to the k -nesting case. Suppose that $w^{(i)} = a_1 a_2 \cdots a_t$ contains a strictly decreasing subsequence $a_{i_1} \cdots a_{i_k}$ of length k , where $1 \leq i_1 < \cdots < i_k \leq t$. By the construction of ϕ and the construction of the sequence $(w^{(0)}, w^{(1)}, \dots, w^{(2n)})$, we deduce that the vertices a_{i_1}, \dots, a_{i_k} are left-hand endpoints of P . For $1 \leq s \leq k$, let b_{j_s} be the right-hand endpoint connected to a_{i_s} . Again, by the construction of $(w^{(0)}, w^{(1)}, \dots, w^{(2n)})$, we see that

$$b_{j_1} > b_{j_2} > \cdots > b_{j_k}. \quad (4.1)$$

Hence the arcs $(a_{i_1}, b_{j_1}), \dots, (a_{i_k}, b_{j_k})$ form a k -crossing of P .

On the other hand, suppose that P has a k -crossing consisting of arcs

$$(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k),$$

where $i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$. By the construction of the sequence $(w^{(0)}, w^{(1)}, \dots, w^{(2n)})$, it is easily checked that $i_1 i_2 \cdots i_k$ forms a strictly decreasing subword of $w^{(2j_1-1)}$. This completes the proof. \blacksquare

Conjecture 1.1 and Conjecture 1.2 are consequences of Theorem 4.1. Like the symmetric joint distribution of the crossing number and the nesting number of ordinary

partitions due to Chen, Deng, Du, Stanley and Yan [3], we restrict our attention to fixed sets of the left-hand endpoints and the right-hand endpoints of a linked partition. For two subsets S and T of $[n]$, let $L_n(S, T)$ be the set of linked partitions of $[n]$ such that S is the set of left-hand endpoints and T is the set of right-hand endpoints. Note that $L_n(S, T)$ may be empty.

Let $f_{n,S,T}(i, j)$ be the number of linked partitions P in $L_n(S, T)$ with $\text{cr}(P) = i$ and $\text{ne}(P) = j$. We have the following symmetry property.

Theorem 4.2 *Let S and T be two subsets of $[n]$. For any positive integers i and j , we have*

$$f_{n,S,T}(i, j) = f_{n,S,T}(j, i).$$

To prove Theorem 4.2, we establish an involution on the set $L_n(S, T)$ that exchanges the crossing number and the nesting number of a linked partition. To this end, we define the conjugate of a Hecke diagram as the transpose of the diagram.

Proof of Theorem 4.2. Taking the conjugate of every Hecke diagram leads to an involution on vacillating Hecke tableaux of empty shape and length $2n$. This yields an involution, denoted ψ , on the set of linked partitions of $[n]$. By Theorem 4.1, we find that ψ exchanges the crossing number and the nesting number of a linked partition. It remains to show that ψ preserves the left-hand endpoints and the right-hand endpoints of a linked partition.

Let P be a linked partition of $[n]$, and let $(\lambda^0, \lambda^1, \dots, \lambda^{2n})$ be the corresponding vacillating Hecke tableau. By the construction of ϕ , we observe that a vertex i of P is a left-hand endpoint if and only if λ^{2i} has at least one more square than λ^{2i-1} , and it is a right-hand endpoint if and only if $\lambda^{2i-2} \neq \lambda^{2i-1}$. Hence the involution ψ preserves the left-hand and the right-hand endpoints. Restricting ψ to $L_n(S, T)$ gives an involution on $L_n(S, T)$. This completes the proof. ■

Here is an example for the involution ψ . Let P be the linked partition given in Figure 4.1. Then $\psi(P)$ is the linked partition in Figure 4.2.

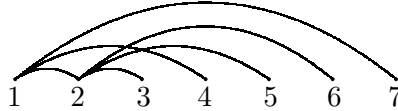


Figure 4.2: An example for the involution ψ .

To conclude, we note that Conjecture 1.1 follows from Theorem 4.2. In fact, Theorem 4.2 also implies Conjecture 1.2. Let P be the linear representation of a linked partition of $[n]$, and let S and T be the sets of left-hand endpoints and right-hand endpoints of P , respectively. It can be seen that P is the front representation of a set partition of $[n]$

if and only if $S \cap T = \emptyset$. Hence the set of front representations of partitions of $[n]$ is the disjoint union of $L_n(S, T)$, where (S, T) ranges over pairs of disjoint subsets of $[n]$. For any two subsets S and T of $[n]$ with $S \cap T = \emptyset$, if $L_n(S, T)$ is not empty, then we can apply Theorem 4.2 to deduce that the crossing number and the nesting number have a symmetric joint distribution over $L_n(S, T)$. Thus we have proved Conjecture 1.2.

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